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FREE VIBRATIONS OF NON-HOMOGENEOUS CIRCULAR AND ANNULAR MEMBRANES

M. Jabareen and M. Eisenberger

Faculty of Civil Engineering, Technion City, Haifa 32000, *Israel.*
E-mail: cvrmosh@techunix.technion.ac.il

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A non-homogeneous membrane is a membrane that has variable thickness or material density. Several recent publications deal with the axisymmetric and antisymmetric modes of transverse vibration for a composite doubly connected and solid membrane with constant or variable density. In this paper, exact solutions for both the axisymmetric and antisymmetric modes of circular and annular membranes with any piecewise polynomial variation of the density are given using a power series solution. The dynamic problem is solved exactly using a recurrence relationship up to any accuracy desired, by deriving the dynamic stiffness matrix for circular and annular membrane elements. Many results for linear, parabolic, and cubic variation of complete and annular membranes are solved using the dynamic stiffness method and presented in the Tables.

1. INTRODUCTION

A non-homogeneous membrane is a membrane that has variable thickness or material density. Several recent publications deal with the axisymmetric and antisymmetric modes of transverse vibration for a composite doubly connected and solid membrane with constant or variable density. Zitnan [1] used a rather complicated discrete least-squares, technique to solve for the natural frequencies of solid circular membranes. Laura *et al.* [2] used Rayleigh–Ritz and differential quadrature solutions for circular and annular membranes with linear variation in density, and compared the results. They solved only for the axisymmetric modes and their results are approximate. Wang [3] gave exact solutions for the vibration frequencies with inverse square density variation. Gutierrez *et al.* [4] gave solutions for the vibration frequencies of linear, parabolic and cubic variations of density using four approximate methods: optimized Rayleigh–Ritz, differential quadrature, finite elements, and lower-bound solution based on the Stodola–Vianello method. They solved only for the axisymmetric modes, and got similar results using all the methods. Laura *et al.* [5] gave an exact solution for the axisymmetric vibration modes of piecewise constant density circular and annular membranes. Rossit *et al.* [6] expanded on the previous solution and added the vibration frequencies for the antisymmetric modes for doubly connected membranes. Buchanan and Peddieson [7] recently presented finite element results for linear variation of density. Their approximate results are accurate as they used 200 degrees of freedom for the eigenvalue problem using a special three-node element.

In this paper, exact solutions for both the axisymmetric and antisymmetric modes of circular and annular membranes with any polynomial variation of the density are given using a power series solution. The dynamic problem is solved exactly using a recurrence

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relationship up to any accuracy desired. Many results for linear, parabolic, and cubic variation of complete and annular membranes are presented in the tables. The mode shapes for several cases are plotted. These exact results can serve as benchmark problems for comparison with approximate methods for assessing their accuracy.

2. GOVERNING EQUATIONS

The governing partial differential equation for the transverse vibrations of circular membranes is

$$S \nabla^2 w(r, \theta, t) = \rho(r) \frac{\partial^2 w(r, \theta, t)}{\partial t^2}, \tag{1}$$

where w is the transverse deflection, S is the tensile force per unit length of the membrane, and ρ is the mass per unit area of the membrane. Using the separation of variables method

$$w(r, \theta, t) = \bar{w}(r) \mathrm{e}^{in\theta} \mathrm{e}^{\mathrm{i}\omega_n t}, \quad n = 0, 1, 2, \ldots \tag{2}$$

and substituting equation (2) into the differential equation (1) gives

$$\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}\bar{w}}{\mathrm{d}r} + \left[\frac{\rho(r)\omega_n^2}{S} - \frac{n^2}{r^2} \right] \bar{w} = 0, \tag{3}$$

To simplify equation (3) the non-dimensional variables

$$r = R_2 \xi, \qquad \bar{w} = R_2 W \tag{4, 5}$$

is introduced, where R_2 is the outer radius of the membrane. The variation of the density can be written as

$$\rho(r) = \rho_0 f(\xi). \tag{6}$$

The function $f(\xi)$ is any polynomial function with known coefficient so that

$$f(\xi) = \sum_{i=0}^{l} f_i \xi^i. \tag{7}$$

Substituting equations (4) and (5) into equation (3) yields

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} + \frac{1}{\xi} \frac{\mathrm{d}W}{\mathrm{d}\xi} + \left[\Omega_n^2 f(\xi) - \frac{n^2}{\xi^2} \right] W = 0 \tag{8}$$

with

$$\Omega_n^2 = \frac{\rho_0 R_2^2 \omega_n^2}{S} \tag{9}$$

3. METHOD OF SOLUTION

To solve equation (8), the Forbenius method will be used. Assuming that

$$W(\xi) = \sum_{i=0}^{\infty} W_i \xi^{i+m}, \tag{10}$$

where m is a constant to be determined from the indicial equation. Then equation (8) becomes

$$\sum_{i=0}^{\infty} W_i\,[(i+m)^2 - n^2]\,\xi^{i+m-2} + \Omega_n^2 \sum_{i=0}^{\infty}\left[\sum_{j=0}^{i} W_{i-j} f_j\right]\xi^{i+m} = 0. \tag{11}$$

Equation (11) can be written as

$$W_0\,[m^2 - n^2]\,\xi^{m-2} + W_1\,[(1+m)^2 - n^2]\xi^{m-1} + \sum_{i=0}^{\infty}\left\{W_{i+2}\,[(i+m+2)^2 - n^2] + \Omega_n^2\left[\sum_{j=0}^{i} W_{i-j} f_j\right]\right\}\xi^{i+m} = 0. \tag{12}$$

From equation (12) the indicial equation can be obtained by zeroing the coefficient of low power in the series $(m-2)$

$$m = \pm\, n. \tag{13}$$

For these two values of m in equation (12) W_0 can be arbitrary, $W_1 = 0$ and the recursive form is

$$W_{i+2} = -\frac{\Omega_n^2 \sum_{j=0}^{i} W_{i-j} f_j}{(i+m+2)^2 - n^2}. \tag{14}$$

The term for W_{i+2} converges to zero as $i \to \infty$, and this is a necessary condition for the convergence of the series.

The suggested solution of equation (10) can be written as a linear combination of two terms. One is related to the first root of the indicial equation $m = +\,n$, and the other to the second root of the indicial equation $m = -\,n$ and to the first term, as will be shown later.

3.1. SOLUTION FOR THE FIRST ROOT $m = n$

The recurrence form of this solution can be derived by substituting $m = n$ into equation (14) such that

$$W_{i+2} = -\frac{\Omega_n^2 \sum_{j=0}^{i} W_{i-j} f_j}{(i+2)(i+2n+2)}, \tag{15}$$

where the first two coefficients are $W_0 = 1$, $W_1 = 0$ and the form of the solution is

$$W_1(\xi) = \sum_{i=0}^{\infty} W_i\,\xi^{i+n}. \tag{16}$$

3.2. SOLUTION FOR THE SECOND ROOT $m = -\,n$

In this solution two cases will be discussed. The first case refers to $n = 0$ and the second refers to $n \geqslant 1$. For $n = 0$, if the recurrence form in equation (14) is used, one independent solution is obtained, so in order to obtain the second independent solution it will be assumed to be in the form

$$W_2(\xi) = W_1(\xi, n=0)\ln(\xi) + \sum_{i=0}^{\infty} W_i'\,\xi^i, \tag{17}$$

where W_i' are unknown coefficients. The coefficients can be found by substituting equation (17) into equation (8),

$$2\sum_{i=0\cdot}^{\infty} W_i\, i\, \xi^{i-2} + \sum_{i=0}^{\infty} W_i'\, i(i-1)\, \xi^{i-2} + \sum_{i=0}^{\infty} W_i'\, i\, \xi^{i-2} + \Omega_n^2 \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} W_{i-j}' f_j\right)\xi^i = 0. \tag{18}$$

Simplifying equation (18), yields

$$\sum_{i=0\cdot}^{\infty}\left\{2W_{i+2}(i+2) + W_{i+2}'(i+2)^2 + \Omega_n^2 \sum_{j=0}^{i} W_{i-j}' f_j\right\}\xi^i = 0. \tag{19}$$

From equation (19) the recurrence from of the second term in the second independent solution can be obtained as,

$$W_{i+2}' = -\frac{\Omega_n^2\left(\sum_{j=0}^{i} W_{i-j}' f_j\right) + 2W_{i+2}(i+2)}{(i+2)^2}, \tag{20}$$

where the first two coefficients are $W_0' = 1$, $W_1' = 0$.

For the case $n \geqslant 1$ it can also be shown that the second solution that is obtained from equation (14) by substituting $m = -n$ is dependent on the first solution, so it is assumed that the second solution can be written as

$$W_2(\xi) = aW_1(\xi)\ln(\xi) + \sum_{i=0}^{\infty} W_i'\, \xi^{i-n}, \tag{21}$$

where a is an unknown constant and W_i' are unknown coefficients.

Substituting equation (21) into equation (8), yields

$$\begin{aligned} 2a\sum_{i=0}^{\infty} W_i(i+n)\,\xi^{i+n-2} &+ \sum_{i=0}^{\infty} W_i'(i-n)(i-n-1)\,\xi^{i-n-2} + \sum_{i=0}^{\infty} W_i'(i-n)\,\xi^{i-n-2} \\ &+ \Omega_n^2 \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} W_{i-j}' f_j\right)\xi^{i-n} - \sum_{i=0}^{\infty} W_i'\,\xi^{i-n-2} = 0. \end{aligned} \tag{22}$$

Simplifying equation (22), yields

$$\begin{aligned} &\sum_{i=0}^{\infty}\left[2a\, W_i(i+n) + W_{i+2n}'(i+2n)(i) + \Omega_n^2 \sum_{j=0}^{i+2n-2} W_{i+2n-2-j}' f_j\right]\xi^{i+n-2} \\ &+ \sum_{i=0}^{2n-3}\left[W_{i+2}'(i+2)(i-2n+2) + \Omega_n^2 \sum_{j=0}^{i} W_{i-j}' f_j\right]\xi^{i-n} = 0 \end{aligned} \tag{23}$$

From equation (23) the recurrence form of the second term in the second independent solution is

$$W_{i+2}' = \begin{cases} -\dfrac{\Omega_n^2 \sum_{j=0}^{i} W_{i-j}' f_j}{(i-2n+2)(i+2)}, & i = 0, \ldots, 2n-3, \\ 1, & i = 2n-2, \\ -\dfrac{\Omega_n^2 \sum_{j=0}^{i} W_{i-j}' f_j + 2(i-n+2)aW_{i-2n+2}}{(i-2n+2)(i+2)}, & i = 2n-1, \ldots, \infty \end{cases} \tag{24}$$

and the constant a is

$$a = -\frac{\Omega_n^2 \sum_{j=0}^{2n-2} W'_{2n-2-j} f_j}{2n} \tag{25}$$

where the first two coefficients are $W'_0 = 1$ and $W'_1 = 0$.

3.3. BOUNDARY CONDITIONS

3.3.1. *Solid circular membrane*

For a circular membrane the second solution has a singularity in the centre of the circle, so the solution

$$W(\xi) = AW_1(\xi) \tag{24'}$$

will be neglected. The recursive form for this solution had been derived in equation (15). Using the boundary condition at $\xi = 1$, the outer radius, the constant A can be determined as

$$W(1) = AW_1(1) = 0. \tag{25'}$$

3.3.2. *Annular circular membrane*

The solution of the annular membrane can be written as

$$W(\xi) = AW_1(\xi) + BW_2(\xi). \tag{26}$$

To determine the constants, two boundary conditions will be implemented at the inner radius, $\xi = \xi_0$, and at the outer radius, $\xi = 1$, so that

$$W(\xi_0) = AW_1(\xi_0) + BW_2(\xi_0) = 0, \qquad W(1) = AW_1(1) + BW_2(1) = 0 \tag{27}$$

and the two equations for A and B_x will be solved.

3.4. FREQUENCY CALCULATIONS

For both cases, the natural frequencies of vibration will be the values of the frequency that yields non-trivial solution to equations (25) and (27). The mode shapes will be found by giving constant A an arbitrary value and then, for annular membranes, finding the value of the dependent constant B. A simple search routine is employed to find the natural frequencies.

3.5. STIFFNESS MATRIX DERIVATION

The derivation of the stiffness matrix is helpful for the solution of multiple-connected regions of different properties, as all the continuity conditions are automatically satisfied at the nodes. The dynamic stiffness matrix contains the forces that are needed to produce unit displacements at the degrees of freedom of the element. For a complete circular membrane element there is only one degree of freedom, the displacement of the outer edge vertically. For an annular membrane element there are two degrees of freedom, the vertical displacements at both the inner and outer edges of the element. Accordingly, the stiffness matrix will be of size (1×1) and (2×2) for complete and annular membranes respectively.

3.5.1. *Solid membrane element*

By using the solution for complete membrane in equation (24), the dynamical stiffness matrix can be derived by imposing unit displacement at the outer edge

$$W(\xi_1) = AW_1(\xi_1) = 1 \tag{28}$$

and then

$$A = \frac{1}{W_1(\xi_1)} \tag{29}$$

and the stiffness is the vertical force needed at the edge

$$K_{11} = S\left.\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi}\right|_{\xi=\xi_1} = S\frac{W_1'(\xi_1)}{W_1(\xi_1)}. \tag{30}$$

3.5.2. *Annular membrane element*

For the case of annular membrane the solution is available in equation (27) and if unit displacement is applied at the two degrees of freedom, firstly at the inner edge, while the outer edge is held. Then

$$W(\xi_1) = AW_1(\xi_1) + BW_2(\xi_1) = 1, \qquad W(\xi_2) = AW_1(\xi_2) + BW_2(\xi_2) = 0. \tag{31, 32}$$

A and B are

$$A = \frac{W_2(\xi_2)}{W_1(\xi_1)W_2(\xi_2) - W_1(\xi_2)W_2(\xi_1)}, \qquad B = -\frac{W_1(\xi_2)}{W_1(\xi_1)W_2(\xi_2) - W_1(\xi_2)W_2(\xi_1)} \tag{33, 34}$$

and the stiffnesses are

$$K_{11} = -S\left.\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi}\right|_{\xi=\xi_1}, \qquad K_{21} = S\left.\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi}\right|_{\xi=\xi_2}. \tag{35, 36}$$

Secondly, application of unit displacement at the outer edge gives

$$W(\xi_1) = AW_1(\xi_1) + BW_2(\xi_1) = 0, \qquad W(\xi_2) = AW_1(\xi_2) + BW_2(\xi_2) = 1 \tag{37, 38}$$

and the constants A and B are

$$A = -\frac{W_2(\xi_1)}{W_1(\xi_1)W_2(\xi_2) - W_1(\xi_2)W_2(\xi_1)}, \qquad B = \frac{W_1(\xi_1)}{W_1(\xi_1)W_2(\xi_2) - W_1(\xi_2)W_2(\xi_1)} \tag{39, 40}$$

and the stiffnesses are

$$K_{12} = -S\left.\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi}\right|_{\xi=\xi_1}, \qquad K_{22} = S\left.\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi}\right|_{\xi=\xi_2}. \tag{41, 42}$$

It is possible to show that for a solid circular membrane with constant density the dynamic stiffness matrix can be written using Bessel functions so that

$$K_{11} = \mathrm{S}\frac{\mathrm{J}_n'(\Omega\xi_1)}{\mathrm{J}_n(\Omega\xi_1)} \tag{43}$$

and for the annular circular membrane

$$K_{11} = S\frac{J_n(\Omega\xi_2)Y'_n(\Omega\xi_1) - Y_n(\Omega\xi_2)J'_n(\Omega\xi_1)}{J_n(\Omega\xi_1)Y_n(\Omega\xi_2) - J_n(\Omega\xi_2)Y_n(\Omega\xi_1)}, \qquad K_{12} = S\frac{Y_n(\Omega\xi_1)J'_n(\Omega\xi_1) - J_n(\Omega\xi_1)Y'_n(\Omega\xi_1)}{J_n(\Omega\xi_1)Y_n(\Omega\xi_2) - J_n(\Omega\xi_2)Y_n(\Omega\xi_1)}, \tag{44, 45}$$

$$K_{21} = S\frac{Y_n(\Omega\xi_2)J'_n(\Omega\xi_2) - J_n(\Omega\xi_2)Y'_n(\Omega\xi_2)}{J_n(\Omega\xi_1)Y_n(\Omega\xi_2) - J_n(\Omega\xi_2)Y_n(\Omega\xi_1)}, \qquad K_{22} = S\frac{J_n(\Omega\xi_1)Y'_n(\Omega\xi_2) - Y_n(\Omega\xi_1)J'_n(\Omega\xi_2)}{J_n(\Omega\xi_1)Y_n(\Omega\xi_2) - J_n(\Omega\xi_2)Y_n(\Omega\xi_1)}. \tag{46, 47}$$

3.6. FREQUENCY CALCULATIONS

For a particular case, the stiffness matrix for the degrees of freedom in the structure is assembled, as in the regular finite element method. The natural frequencies of vibration will be the values of frequency that cause the determinant of the dynamic stiffness to become singular. A simple search routine is employed to find the natural frequencies.

4. NUMERICAL RESULTS

Results for the exact non-dimensional frequencies of circular and annular membranes with several variation parameters and linear, parabolic and cubic density variation are presented in the tables. Tables 1–3 present the values for linear variation. Tables 4–6 present the values for parabolic variation and cubic variations are given in Tables 7–9. The first five

TABLE 1

Values of Ω for $R_1/R_2 = 0{\cdot}0$ and $f(\xi) = 1 + \alpha\xi$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	2·4048	2·1827	2·0108	1·8731	1·7598	1·6646	1·5832
Ω_{02}	5·5201	4·9629	4·5549	4·2374	3·9802	3·7659	3·5835
Ω_{03}	8·6537	7·7659	7·1196	6·6190	6·2150	5·8792	5·5940
Ω_{04}	11·7915	10·5746	9·6901	9·0059	8·4543	7·9964	7·6078
Ω_{05}	14·9309	13·3856	12·2631	11·3951	10·6958	10·1155	9·6233
Ω_{11}	3·8317	3·3887	3·0678	2·8224	2·6273	2·4674	2·3334
Ω_{12}	7·0156	6·2537	5·6991	5·2708	4·9268	4·6425	4·4023
Ω_{13}	10·1735	9·0895	8·3006	7·6905	7·1994	6·7924	6·4479
Ω_{14}	13·3237	11·9160	10·8920	10·0999	9·4618	8·9327	8·4844
Ω_{15}	16·4706	14·7383	13·4787	12·5043	11·7192	11·0680	10·5160
Ω_{21}	5·1356	4·4808	4·0224	3·6795	3·4110	3·1936	3·0129
Ω_{22}	8·4172	7·4442	6·7454	6·2119	5·7874	5·4392	5·1470
Ω_{23}	11·6198	10·3284	9·3944	8·6764	8·1016	7·6279	7·2285
Ω_{24}	14·7960	13·1845	12·0159	11·1150	10·3919	9·7943	9·2895
Ω_{25}	17·9598	16·0269	14·6236	13·5404	12·6696	11·9490	11·3395
Ω_{31}	6·3802	5·5186	4·9284	4·4949	4·1550	3·8831	3·6582
Ω_{32}	9·7610	8·5773	7·7383	7·1041	6·6032	6·1948	5·8536
Ω_{33}	13·0152	11·5137	10·4363	9·6137	8·9588	8·4216	7·9704
Ω_{34}	16·2235	14·4037	13·0906	12·0831	11·2776	10·6145	10·0560
Ω_{35}	19·4094	17·2703	15·7224	14·5313	13·5768	12·7892	12·1247

TABLE 2

Values of Ω for $R_1/R_2 = 0{\cdot}2$ and $f(\xi) = 1 + \alpha\xi$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	3·8160	3·3470	3·0156	2·7659	2·5691	2·4090	2·2754
Ω_{02}	7·7855	6·8319	6·1640	5·6615	5·2654	4·9426	4·6728
Ω_{03}	11·7321	10·2972	9·2937	8·5391	7·9443	7·4594	7·0540
Ω_{04}	15·6702	13·7550	12·4163	11·4098	10·6164	9·9695	9·4288
Ω_{05}	19·6042	17·2091	15·5354	14·2771	13·2852	12·4765	11·8005
Ω_{11}	4·2357	3·7069	3·3349	3·0556	2·8360	2·6577	2·5092
Ω_{12}	8·0554	7·0707	6·3802	5·8603	5·4504	5·1162	4·8369
Ω_{13}	11·9266	10·4716	9·4532	8·6870	8·0826	7·5898	7·1778
Ω_{14}	15·8210	13·8912	12·5416	11·5264	10·7259	10·0731	9·5273
Ω_{15}	19·7271	17·3205	15·6381	14·3730	13·3754	12·5619	11·8818
Ω_{21}	5·2218	4·5442	4·0734	3·7230	3·4492	3·2279	3·0443
Ω_{22}	8·8039	7·7281	6·9718	6·4021	5·9526	5·5863	5·2802
Ω_{23}	12·4936	10·9787	9·9156	9·1144	8·4817	7·9655	7·5336
Ω_{24}	16·2683	14·2948	12·9123	11·8710	11·0492	10·3785	9·8175
Ω_{25}	20·0935	17·6527	15·9445	14·6588	13·6443	12·8166	12·1242
Ω_{31}	6·3946	5·5284	4·9359	4·4991	4·1603	3·8878	3·6625
Ω_{32}	9·8739	8·6532	7·7956	7·1505	6·6425	6·2291	5·8842
Ω_{33}	13·3807	11·7648	10·6271	9·7681	9·0892	8·5349	8·0712
Ω_{34}	16·9940	14·9474	13·5096	12·4245	11·5669	10·8664	10·2800
Ω_{35}	20·6968	18·1994	16·4481	15·1279	14·0850	13·2334	12·5206

TABLE 3

Values of Ω for $R_1/R_2 = 0{\cdot}5$ and $f(\xi) = 1 + \alpha\xi$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	6·2461	5·3259	4·7198	4·2822	3·9473	3·6803	3·4610
Ω_{02}	12·5469	10·7022	9·4894	8·6138	7·9433	7·4085	6·9690
Ω_{03}	18·8364	16·0683	14·2489	12·9355	11·9295	11·1271	10·4677
Ω_{04}	25·1228	21·4314	19·0056	17·2543	15·9130	14·8430	13·9636
Ω_{05}	31·4079	26·7934	23·7611	21·5720	19·8953	18·5578	17·4585
Ω_{11}	6·3932	5·4506	4·8298	4·3818	4·0389	3·7657	3·5412
Ω_{12}	12·6247	10·7690	9·5488	8·6678	7·9932	7·4551	7·0129
Ω_{13}	18·8889	16·1134	14·2892	12·9722	11·9635	11·1588	10·4975
Ω_{14}	25·1624	21·4655	19·0360	17·2820	15·9386	14·8670	13·9862
Ω_{15}	31·4405	26·8208	23·7855	21·5942	19·9159	18·5770	17·4766
Ω_{21}	6·8138	5·8069	5·1443	4·6663	4·3007	4·0093	3·7701
Ω_{22}	12·8555	10·9670	9·7250	8·8281	8·1412	7·5933	7·1430
Ω_{23}	19·0457	16·2483	14·4095	13·0817	12·0648	11·2535	10·5867
Ω_{24}	25·2808	21·5675	19·1270	17·3649	16·0153	14·9387	14·0538
Ω_{25}	31·5351	26·9026	23·8586	21·6608	19·9775	18·6347	17·5310
Ω_{31}	7·4577	6·3515	5·6245	5·1005	4·6998	4·3808	4·1189
Ω_{32}	13·2318	11·2898	10·0122	9·0894	8·3825	7·8186	7·3551
Ω_{33}	19·3045	16·4711	14·6080	13·2627	12·2321	11·4099	10·7341
Ω_{34}	25·4770	21·7365	19·2778	17·5024	16·1426	15·0576	14·1659
Ω_{35}	31·6927	27·0386	23·9799	21·7715	20·0799	18·7304	17·6212

TABLE 4

Values of Ω for $R_1/R_2 = 0{\cdot}0$ and $f(\xi) = 1 + \alpha\xi^2$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	2·4048	2·2819	2·1736	2·0778	1·9925	1·9162	1·8474
Ω_{02}	5·5201	5·1412	4·8416	4·5999	4·3914	4·2149	4·0608
Ω_{03}	8·6537	8·0406	7·5595	7·1667	6·8369	6·5541	6·3076
Ω_{04}	11·7915	10·9479	10·2878	9·7492	9·2967	8·9085	8·5701
Ω_{05}	14·9309	13·8582	13·0199	12·3360	11·7616	11·2687	10·8387
Ω_{11}	3·8317	3·5429	3·3053	3·1068	2·9385	2·7939	2·6681
Ω_{12}	7·0156	6·5024	6·0969	5·7638	5·4825	5·2399	5·0275
Ω_{13}	10·1735	9·4337	8·8533	8·3787	7·9792	7·6357	7·3354
Ω_{14}	13·3237	12·3569	11·6004	10·9826	10·4631	10·0169	9·6272
Ω_{15}	16·4706	15·2768	14·3437	13·5821	12·9420	12·3924	11·9127
Ω_{21}	5·1356	4·6718	4·3065	4·0115	3·7678	3·5626	3·3870
Ω_{22}	8·4172	7·7513	7·2269	6·7977	6·4371	6·1280	5·8593
Ω_{23}	11·6198	10·7378	10·0457	9·4795	9·0030	8·5936	8·2362
Ω_{24}	14·7960	13·6929	12·8293	12·1235	11·5296	11·1092	10·5734
Ω_{25}	17·9598	16·6336	15·5965	14·7494	14·0370	13·4249	12·8903
Ω_{31}	6·3802	5·7377	5·2470	4·8595	4·5446	4·2829	4·0610
Ω_{32}	9·7610	8·9325	8·2850	7·7592	7·3208	6·9482	6·6263
Ω_{33}	13·0152	11·9808	11·1706	10·5093	9·9544	9·4792	9·0660
Ω_{34}	16·2235	14·9749	13·9975	13·1990	12·5276	11·9515	11·4492
Ω_{35}	19·4094	17·9425	16·7949	15·8574	15·0690	14·3918	13·8009

TABLE 5

Values of Ω for $R_1/R_2 = 0{\cdot}2$ and $f(\xi) = 1 + \alpha\xi^2$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	3·8160	3·4969	3·2431	3·0358	2·8627	2·7157	2·5888
Ω_{02}	7·7855	7·1061	6·5875	6·1730	5·8307	5·5411	5·2917
Ω_{03}	11·7321	10·7020	9·9201	9·2969	8·7833	8·3494	7·9761
Ω_{04}	15·6702	14·2917	13·2474	12·4158	11·7307	11·1523	10·6547
Ω_{05}	19·6042	17·8784	16·5720	15·5320	14·6755	13·9525	13·3307
Ω_{11}	4·2357	3·8703	3·5811	3·3461	3·1507	2·9853	2·8430
Ω_{12}	8·0554	7·3561	6·8215	6·3934	6·0394	5·7395	5·4810
Ω_{13}	11·9266	10·8846	10·0932	9·4618	8·9410	8·5008	8·1217
Ω_{14}	15·8210	14·4344	13·3834	12·5460	11·8559	11·2730	10·7714
Ω_{15}	19·7271	17·9950	16·6835	15·6391	14·7788	14·0524	13·4275
Ω_{21}	5·2218	4·7356	4·3570	4·0534	3·8038	3·5943	3·4154
Ω_{22}	8·8039	8·0439	7·4598	6·9900	6·6004	6·2698	5·9845
Ω_{23}	12·4936	11·4159	10·5949	9·9380	9·3948	8·9346	8·5377
Ω_{24}	16·2683	14·8571	13·7860	12·9312	12·2257	11·6290	11·1149
Ω_{25}	20·0935	18·3428	17·0162	15·9588	15·0870	14·3502	13·7159
Ω_{31}	6·3946	5·7473	5·2541	4·8651	4·5493	4·2869	4·0645
Ω_{32}	9·8739	9·0091	8·3412	7·8028	7·3562	6·9778	6·6518
Ω_{33}	13·3807	12·2394	11·3650	10·6620	10·0785	9·5829	9·1546
Ω_{34}	16·9940	15·5411	14·4347	13·5490	12·8159	12·1944	11·6577
Ω_{35}	20·6968	18·9156	17·5634	16·4838	15·5920	14·8372	14·1863

TABLE 6

Values of Ω for $R_1/R_2 = 0{\cdot}5$ and $f(\xi) = 1 + \alpha\xi^2$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	6·2461	5·5071	4·9782	4·5762	4·2577	3·9974	3·7795
Ω_{02}	12·5469	11·0525	9·9959	9·1963	8·5634	8·0460	7·6126
Ω_{03}	18·8364	16·5904	15·0056	13·8073	12·8590	12·0839	11·4346
Ω_{04}	25·1228	22·1261	20·0130	18·4158	17·1520	16·1189	15·2536
Ω_{05}	31·4079	27·6609	25·0196	23·0233	21·4439	20·1528	19·0714
Ω_{11}	6·3932	5·6355	5·0934	4·6816	4·3554	4·0887	3·8656
Ω_{12}	12·6247	11·1217	10·0589	9·2545	8·6178	8·0972	7·6612
Ω_{13}	18·8889	16·6373	15·0484	13·8470	12·8963	12·1190	11·4680
Ω_{14}	25·1624	22·1615	20·0454	18·4459	17·1802	16·1456	15·2789
Ω_{15}	31·4405	27·6892	25·0456	23·0475	21·4665	20·1742	19·0917
Ω_{21}	6·8138	6·0024	5·4224	4·9822	4·6337	4·3491	4·1110
Ω_{22}	12·8555	11·3271	10·2459	9·4274	8·7793	8·2493	7·8053
Ω_{23}	19·0457	16·7774	15·1764	13·9657	13·0074	12·2240	11·5677
Ω_{24}	25·2808	22·2674	20·1423	18·5358	17·2645	16·2253	15·3547
Ω_{25}	31·5351	27·7743	25·1235	23·1198	21·5344	20·2383	19·1527
Ω_{31}	7·4577	6·5624	5·9236	5·4396	5·0568	4·7445	4·4836
Ω_{32}	13·2318	11·6619	10·5507	9·7091	9·0424	8·4971	8·0401
Ω_{33}	19·3045	17·0087	15·3879	14·1617	13·1911	12·3973	11·7323
Ω_{34}	25·4770	22·4430	20·3031	18·6851	17·4045	16·3575	15·4804
Ω_{35}	31·6927	27·9156	25·2528	23·2400	21·6471	20·3449	19·2541

TABLE 7

Values of Ω for $R_1/R_2 = 0{\cdot}0$ and $f(\xi) = 1 + \alpha\xi^3$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	2·4048	2·3312	2·2620	2·1971	2·1362	2·0793	2·0258
Ω_{02}	5·5201	5·2374	4·9993	4·7969	4·6227	4·4709	4·3370
Ω_{03}	8·6537	8·1891	7·8075	7·4854	7·2076	6·9638	6·7472
Ω_{04}	11·7915	11·1491	10·6250	10·1839	9·8039	9·4708	9·1750
Ω_{05}	14·9309	14·1122	13·4463	12·8864	12·4043	11·9819	11·6068
Ω_{11}	3·8317	3·6371	3·4633	3·3084	3·1699	3·0457	2·9338
Ω_{12}	7·0156	6·6344	6·3191	6·0526	5·8226	5·6206	5·4405
Ω_{13}	10·1735	9·6146	9·1577	8·7726	8·4408	8·1500	7·8917
Ω_{14}	13·3237	12·5890	11·9909	11·4876	11·0542	10·6743	10·3370
Ω_{15}	16·4706	15·5608	14·8216	14·2001	13·6650	13·1963	12·7800
Ω_{21}	5·1356	4·8010	4·5142	4·2674	4·0536	3·8668	3·7023
Ω_{22}	8·4172	7·9241	7·5180	7·1740	6·8760	6·6136	6·3794
Ω_{23}	11·6198	10·9574	10·4166	9·9608	9·5675	9·2223	8·9151
Ω_{24}	14·7960	13·9618	13·2832	12·7121	12·2200	11·7885	11·4050
Ω_{25}	17·9598	16·9530	16·1353	15·4477	14·8556	14·3366	13·8755
Ω_{31}	6·3802	5·8954	5·4926	5·1553	4·8693	4·6238	4·4107
Ω_{32}	9·7610	9·1449	8·6387	8·2103	7·8399	7·5148	7·2261
Ω_{33}	13·0152	12·2408	11·6089	11·0758	10·6155	10·2111	9·8511
Ω_{34}	16·2235	15·2832	14·5183	13·8742	13·3188	12·8313	12·3977
Ω_{35}	19·4094	18·3000	17·3991	16·6412	15·9881	15·4154	14·9062

TABLE 8

Values of Ω for $R_1/R_2 = 0{\cdot}2$ and $f(\xi) = 1 + \alpha\xi^3$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	3·8160	3·5942	3·4029	3·2366	3·0908	2·9620	2·8472
Ω_{02}	7·7855	7·2698	6·8553	6·5122	6·2211	5·9692	5·7478
Ω_{03}	11·7321	10·9413	10·3121	9·7928	9·3530	8·9730	8·6399
Ω_{04}	15·6702	14·6081	13·7658	13·0717	12·4840	11·9766	11·5317
Ω_{05}	19·6042	18·2723	17·2177	16·3490	15·6138	14·9791	14·4226
Ω_{11}	4·2357	3·9781	3·7569	3·5654	3·3984	3·2514	3·1211
Ω_{12}	8·0554	7·5261	7·1006	6·7478	6·4478	6·1878	5·9589
Ω_{13}	11·9266	11·1281	10·4925	9·9678	9·5230	9·1386	8·8013
Ω_{14}	15·8210	14·7537	13·9072	13·2093	12·6184	12·1079	11·6602
Ω_{15}	19·7271	18·3912	17·3333	16·4619	15·7242	15·0873	14·5287
Ω_{21}	5·2218	4·8675	4·5674	4·3115	4·0912	3·8996	3·7313
Ω_{22}	8·8039	8·2324	7·7713	7·3870	7·0583	6·7718	6·5186
Ω_{23}	12·4936	11·6716	11·0165	10·4744	10·0140	9·6150	9·2640
Ω_{24}	16·2683	15·1854	14·3260	13·6169	13·0159	12·4961	12·0397
Ω_{25}	20·0935	18·7460	17·6787	16·7991	16·0541	15·4105	14·8457
Ω_{31}	6·3946	5·9053	5·5000	5·1610	4·8740	4·7278	4·4141
Ω_{32}	9·8739	9·2261	8·6999	8·2582	7·8785	7·5467	7·2530
Ω_{33}	13·3807	12·5162	11·8241	11·2486	10·7573	10·3296	9·9517
Ω_{34}	16·9940	15·8847	15·0027	14·2732	13·6533	13·1159	12·6427
Ω_{35}	20·6968	19·3306	18·2476	17·3541	16·5964	15·9409	15·3650

TABLE 9

Values of Ω for $R_1/R_2 = 0{\cdot}5$ and $f(\xi) = 1 + \alpha\xi^3$

	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}5$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}5$	$\alpha = 2{\cdot}0$	$\alpha = 2{\cdot}5$	$\alpha = 3{\cdot}0$
Ω_{01}	6·2461	5·6512	5·1953	4·8324	4·5353	4·2865	4·0742
Ω_{02}	12·5469	11·3180	10·4015	9·6816	9·0958	8·6062	8·1889
Ω_{03}	18·8364	16·9828	15·6063	14·5274	13·6502	12·9177	12·2936
Ω_{04}	25·1228	22·6466	20·8104	19·3722	18·2033	17·2275	16·3961
Ω_{05}	31·4079	28·3098	26·0143	24·2167	22·7560	21·5367	20·4979
Ω_{11}	6·3932	5·7827	5·3150	4·9429	4·6383	4·3833	4·1657
Ω_{12}	12·6247	11·3891	10·4674	9·7435	9·1542	8·6617	8·2419
Ω_{13}	18·8889	17·0309	15·6511	14·5696	13·6902	12·9559	12·3301
Ω_{14}	25·1624	22·6829	20·8443	19·4042	18·2337	17·2565	16·4239
Ω_{15}	31·4405	28·3393	26·0415	24·2424	22·7804	21·5600	20·5202
Ω_{21}	6·8138	6·1583	5·6564	5·2576	4·9315	4·6586	4·4261
Ω_{22}	12·8555	11·5999	10·6631	9·9269	9·3274	8·8263	8·3990
Ω_{23}	19·0457	17·1748	15·7852	14·6958	13·8099	13·0700	12·4394
Ω_{24}	25·2808	22·7917	20·9458	19·4999	18·3246	17·3433	16·5071
Ω_{25}	31·5351	28·4264	26·1231	24·3193	22·8535	21·6298	20·5872
Ω_{31}	7·4577	6·7312	6·1759	5·7355	5·3760	5·0757	4·8201
Ω_{32}	13·2318	11·9437	10·9821	10·2260	9·6098	9·0944	8·6548
Ω_{33}	19·3045	17·4123	16·0066	14·9043	14·0076	13·2585	12·6199
Ω_{34}	25·4770	22·9720	21·1142	19·6587	18·4753	17·4872	16·6451
Ω_{35}	31·6927	28·5713	26·2586	24·4472	22·9750	21·7459	20·6986

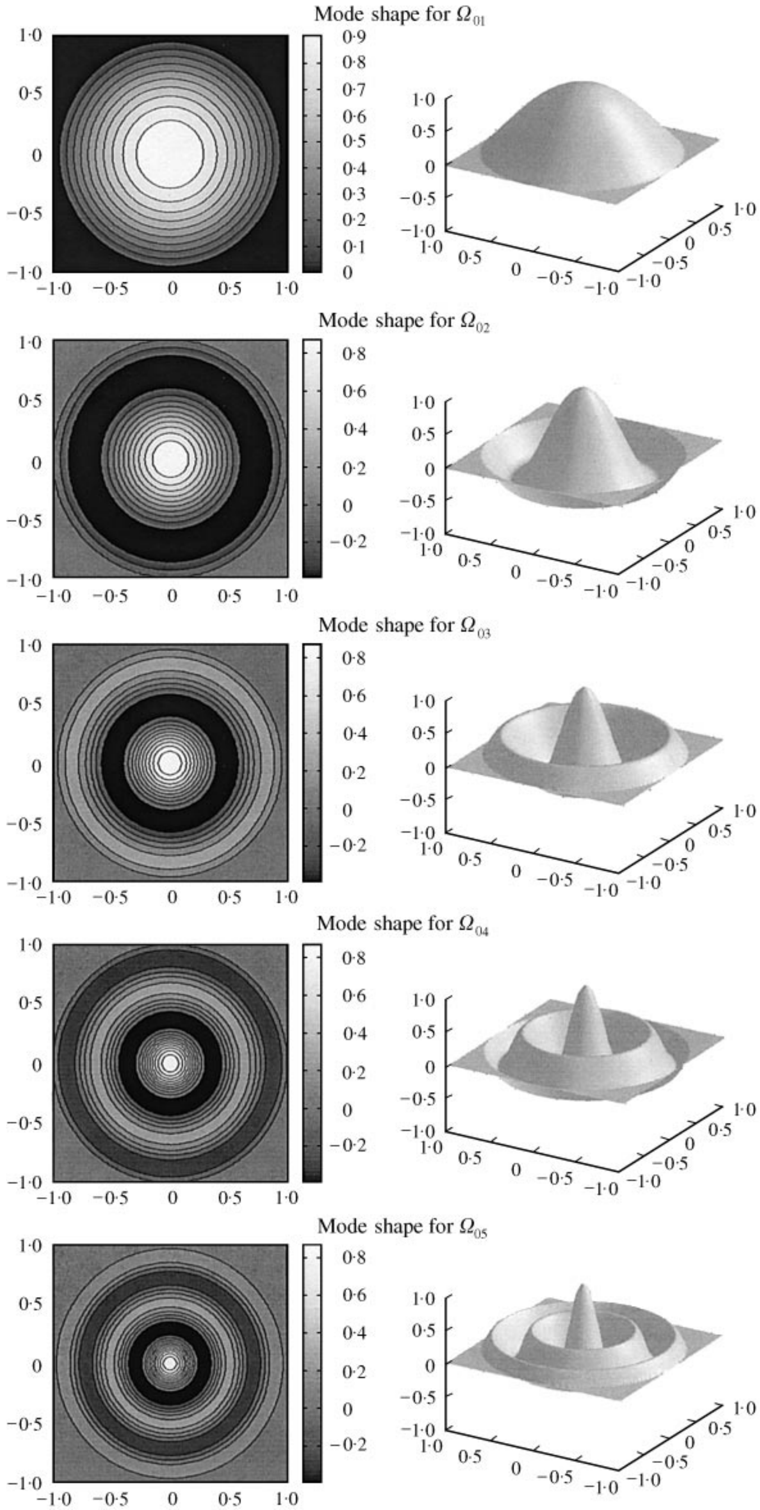

Figure 1. Mode shapes of solid circular membrane for $n = 0$.

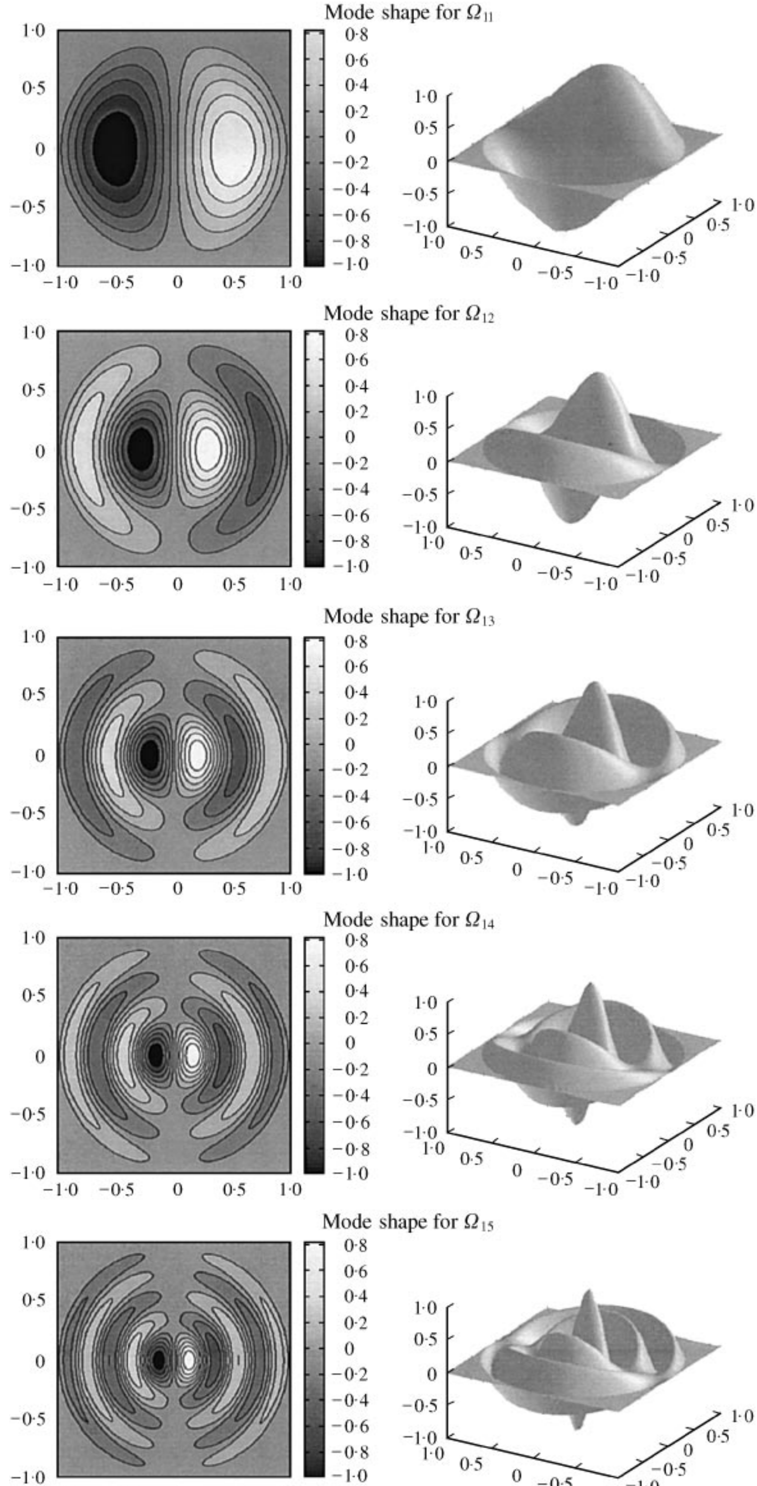

Figure 2. Mode shapes of solid circular membrane for $n = 1$.

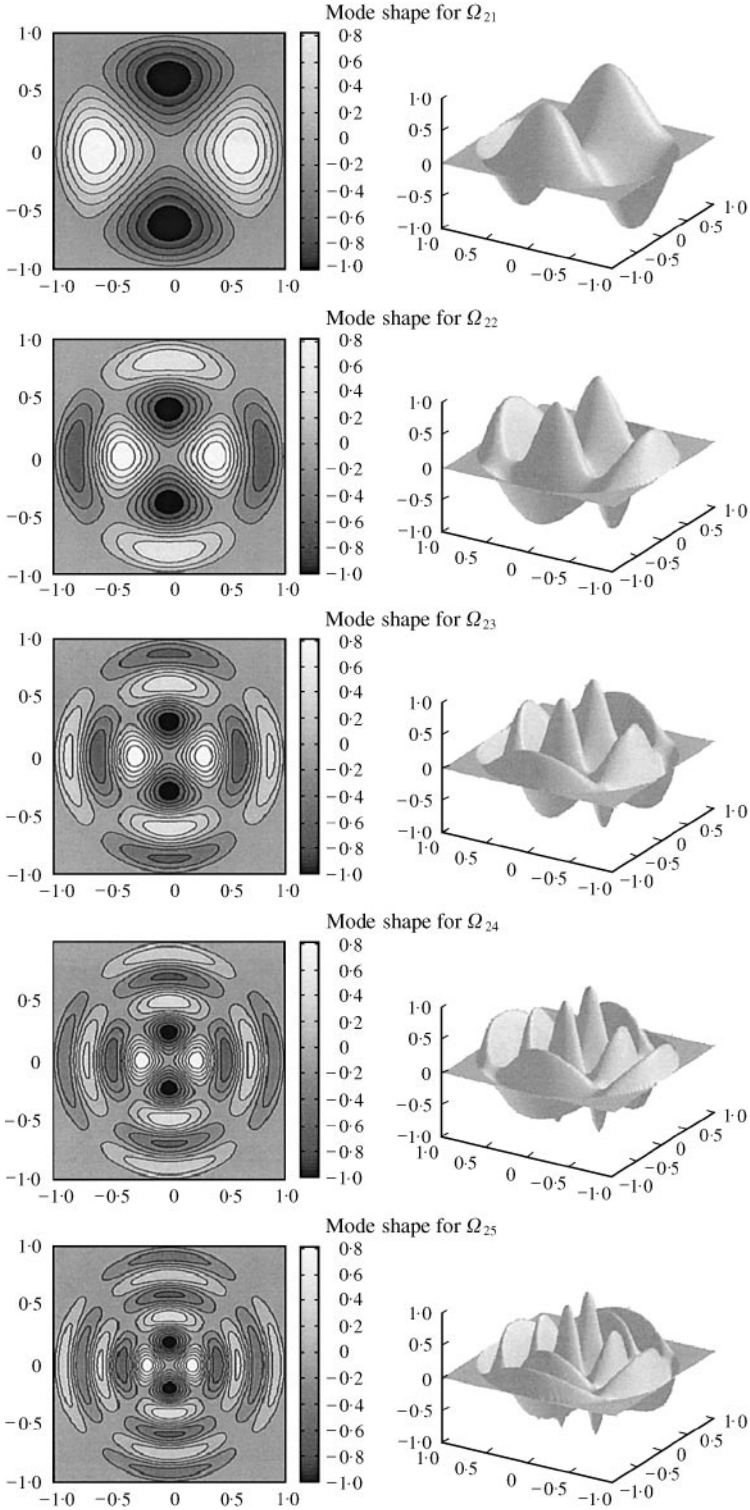

Figure 3. Mode shapes of solid circular membrane for $n = 2$.

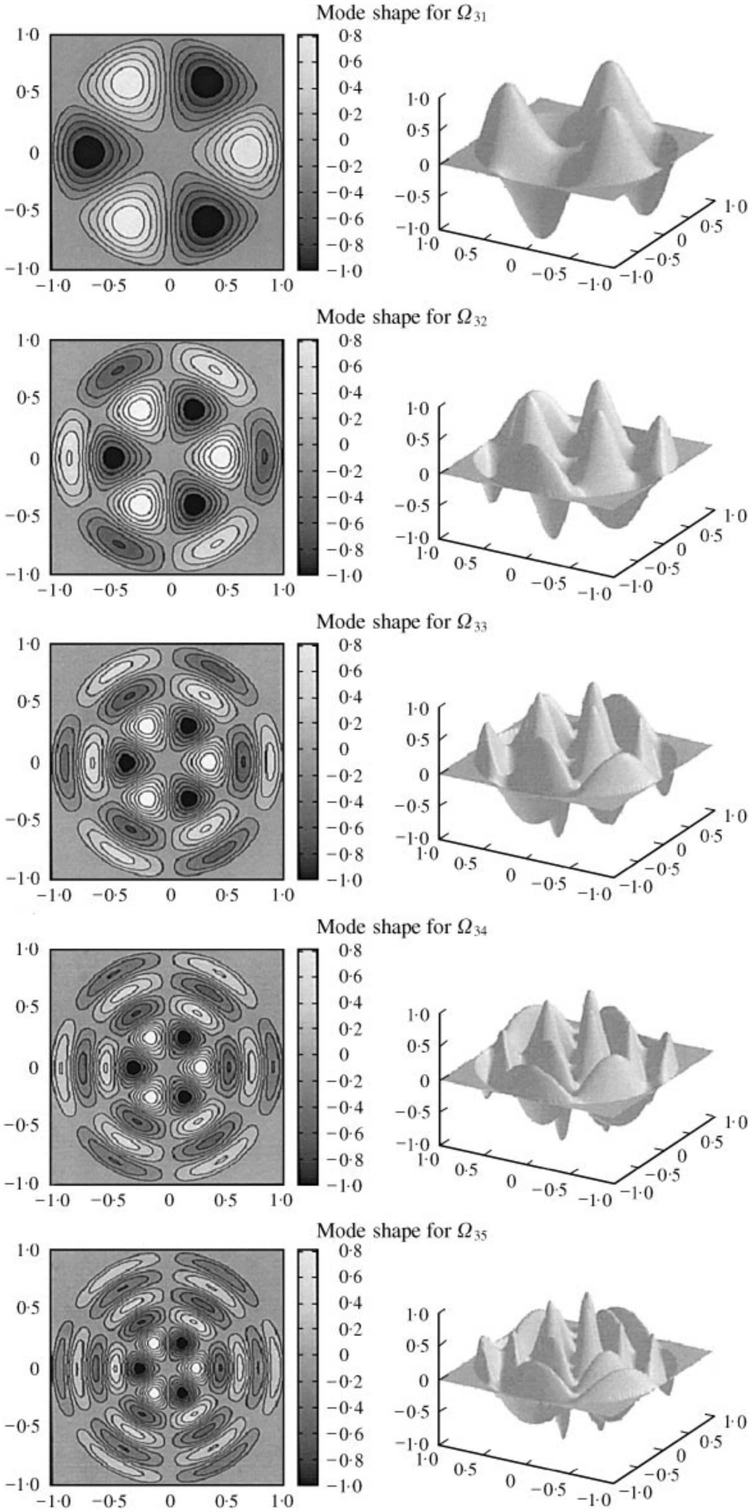

Figure 4. Mode shapes of solid circular membrane for $n = 3$.

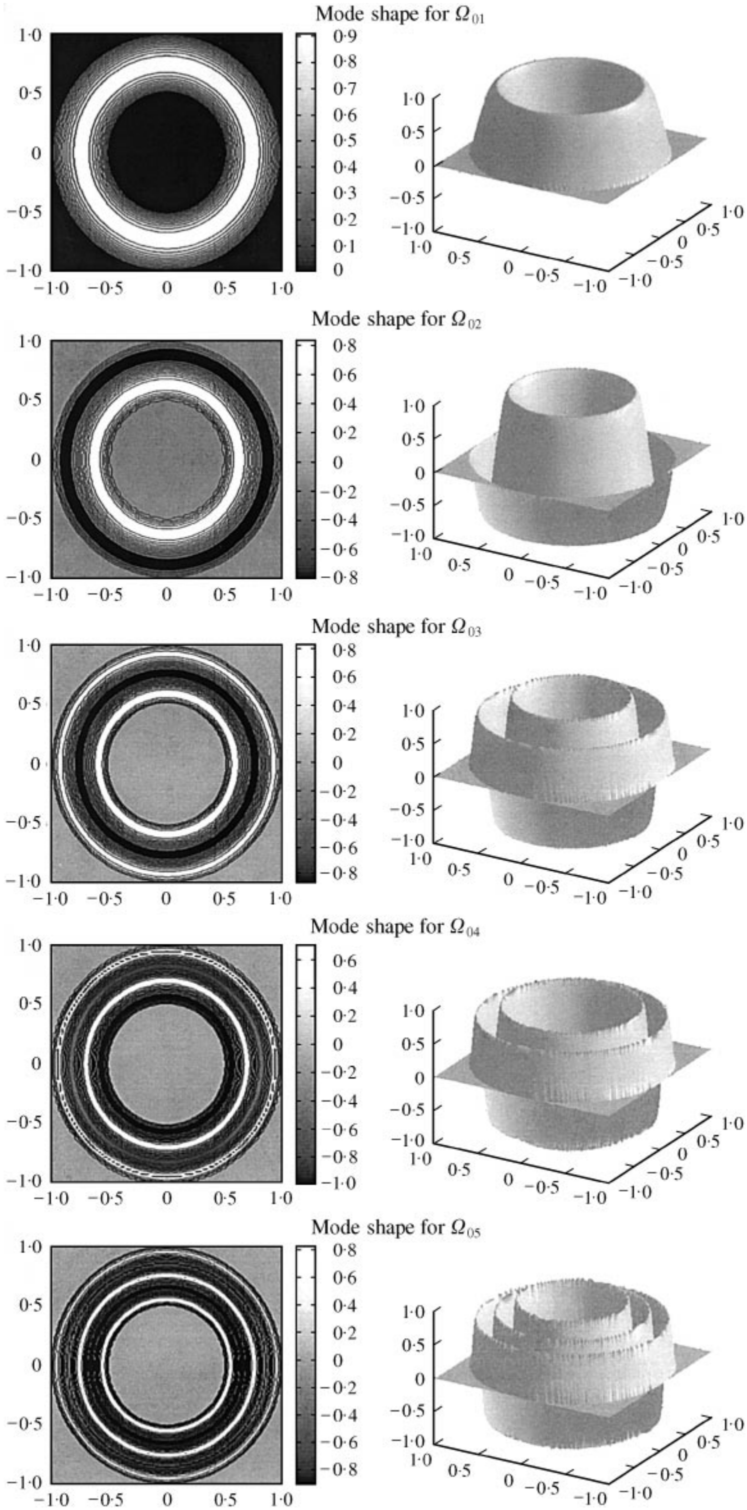

Figure 5. Mode shapes of annular membrane, $R_1/R_2 = 0{\cdot}5$ for $n = 0$.

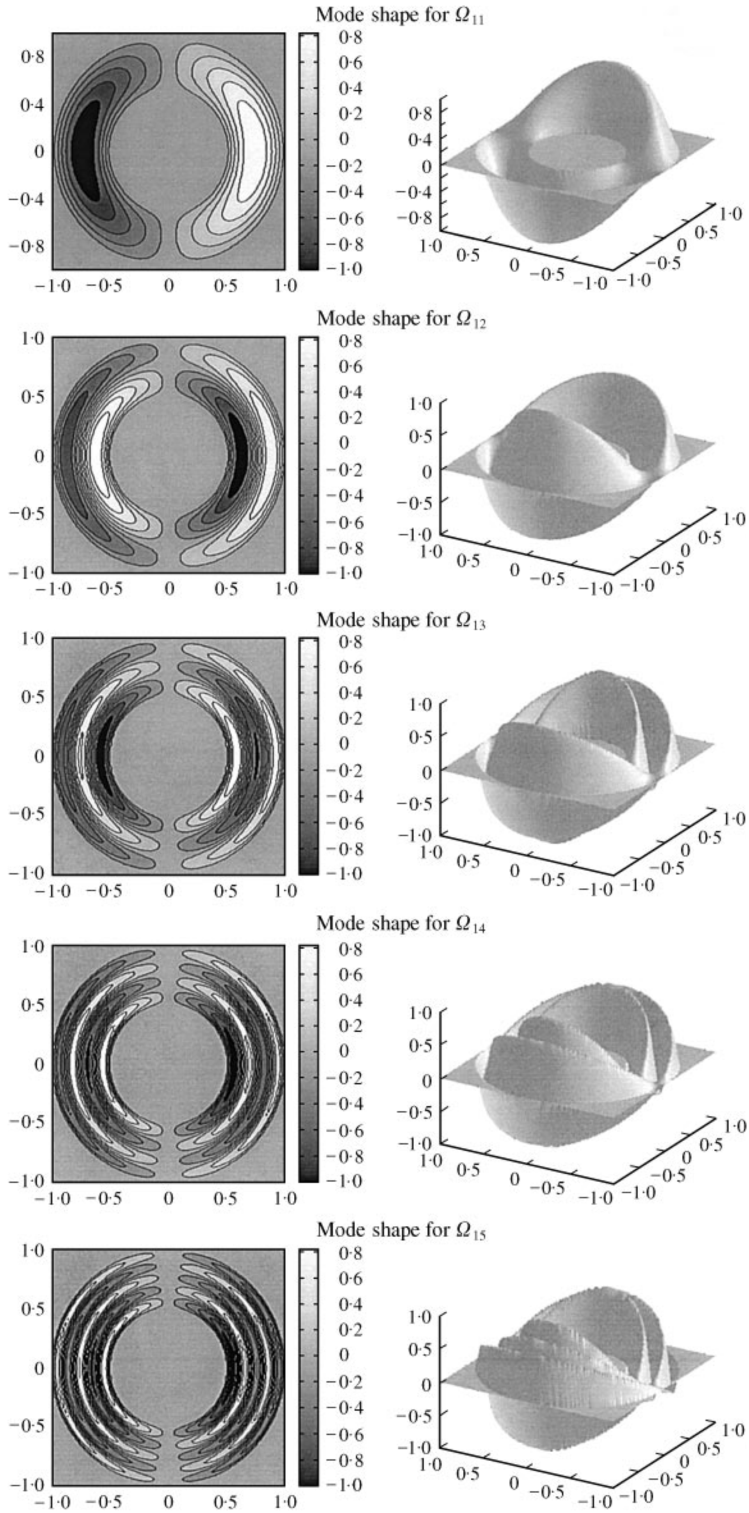

Figure 6. Mode shapes of annular membrane, $R_1/R_2 = 0{\cdot}5$ for $n = 1$.

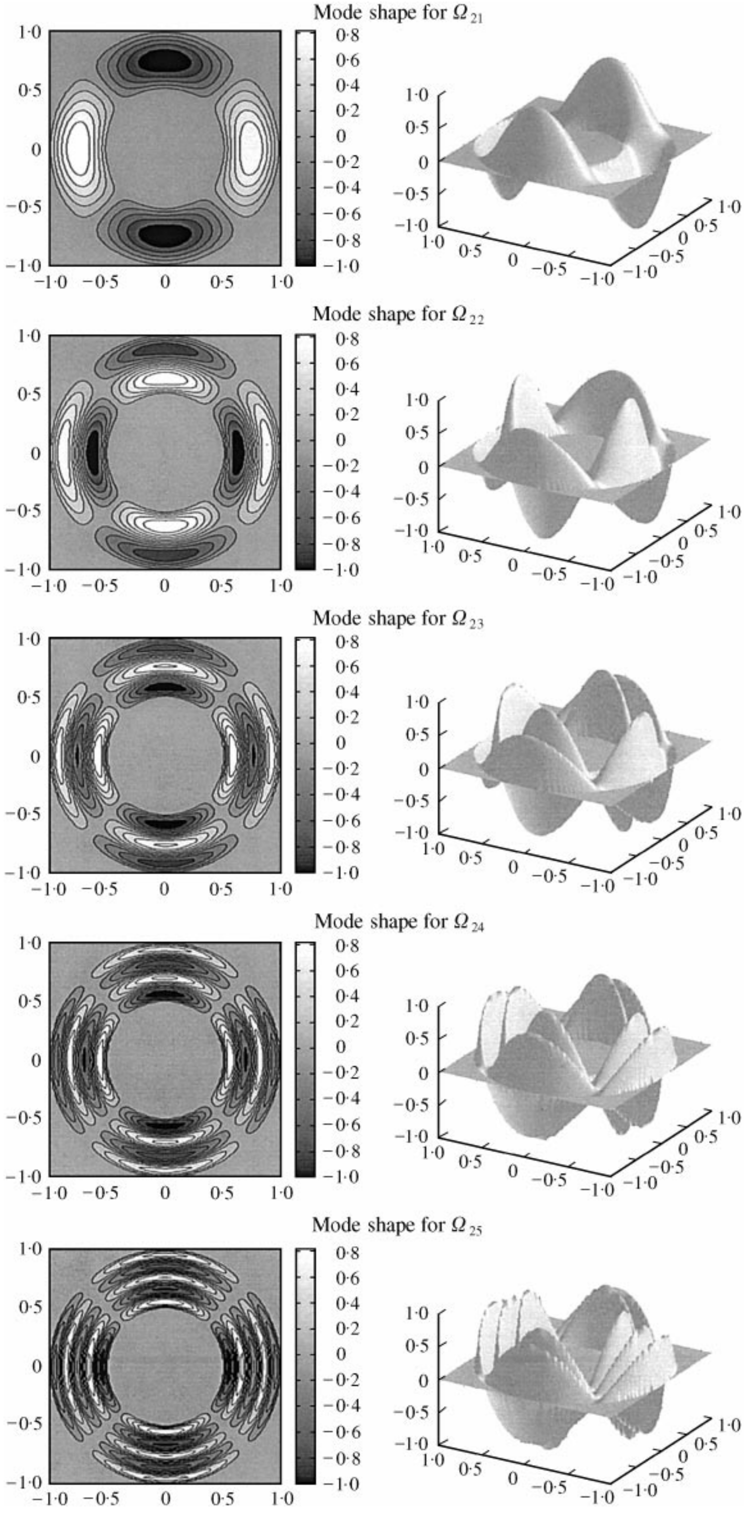

Figure 7. Mode shapes of annular membrane, $R_1/R_2 = 0{\cdot}5$ for $n = 2$.

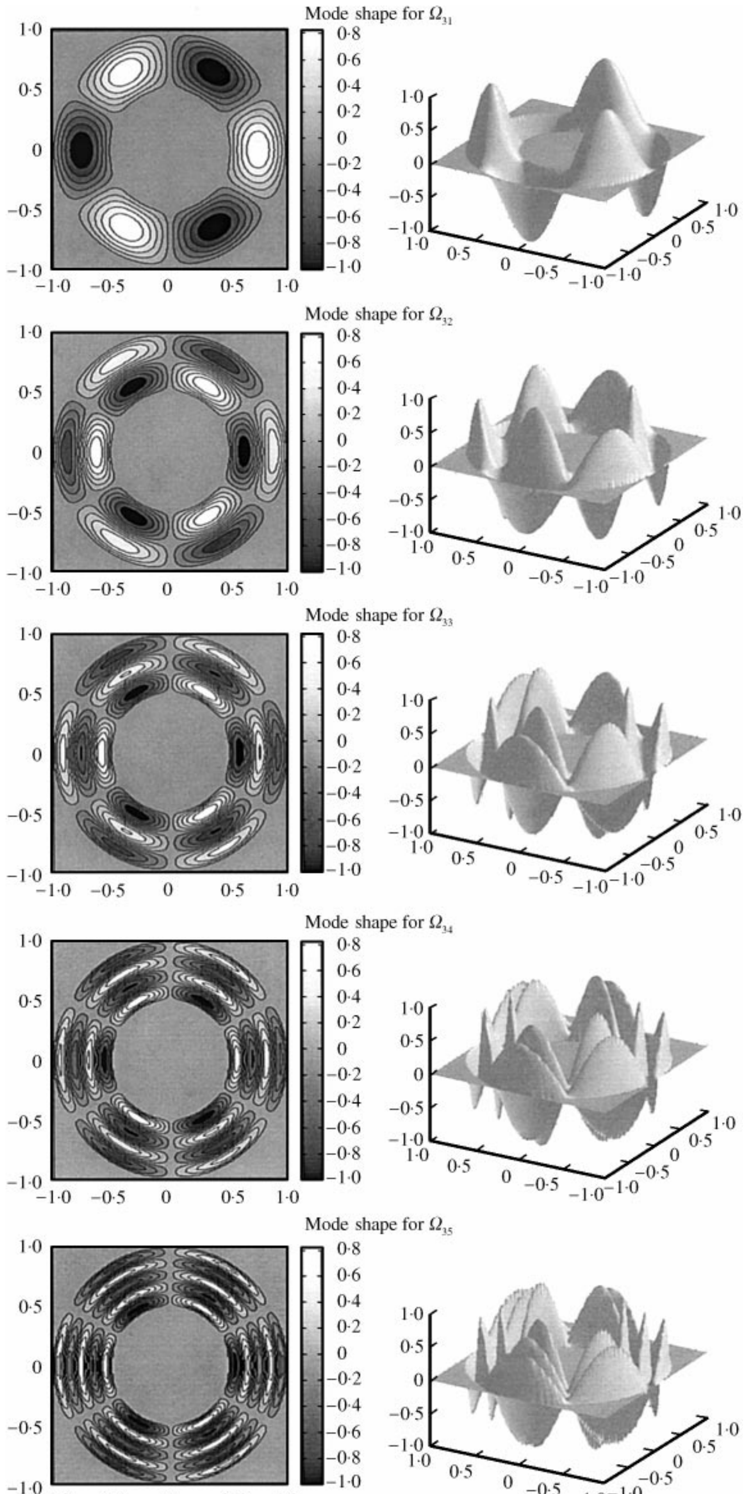

Figure 8. Mode shapes of annular membrane, $R_1/R_2 = 0{\cdot}5$ for $n = 3$.

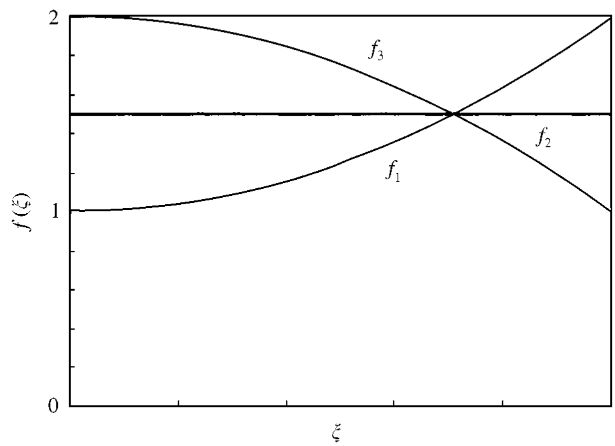

Figure 9. Density variations of example membranes.

normal modes of solid membranes with linear variation in density for axisymmetric, and the first three antisymmetric modes (with 1, 2, and 3 radial nodal lines) are presented in Figures 1–4. In Figures 5–8 the normal modes for annular membrane with the same variation are shown.

The influence of the effect of thickness variation on the natural mode is studied by comparing the results for membranes with constant density and two types of parabolic variations (Figure 9). The results are given in Table 10. All these membranes have the same total mass so the results present the effect of mass distribution: the case of a denser central region (i.e., more mass is located towards the centre) is the more flexible case and the vibration frequencies are the lowest. The other two cases show that moving the mass towards the boundaries in the outer region will cause stiffening of the membrane and higher vibration frequencies.

In the next example the dynamic stiffness method is used to analyze an annular membrane with a density variation $f(\xi)$ which consists of three piecewise-continuous functions,

$$f(\xi) = \begin{cases} 0{\cdot}6 + 4\xi, & 0{\cdot}1 \leqslant \xi \leqslant 0{\cdot}35, \\ 2, & 0{\cdot}35 \leqslant \xi \leqslant 0{\cdot}75, \\ 5 - 4\xi, & 0{\cdot}75 \leqslant \xi \leqslant 1. \end{cases} \tag{48}$$

In this case, the size of the resulting stiffness matrix will be (2×2), and the degrees of freedom will be the displacements at the locations of the abrupt change in the variations at ($\xi = 0{\cdot}35$ and $0{\cdot}75$). The first three non-dimensional frequencies of the axisymmetric case ($n = 0$) and three antisymmetric ($n = 1, 2, 3$) modes are presented in Table 11.

5. CONCLUDING REMARKS

The natural vibration frequencies and mode shapes are found for variable density membranes using both the direct approach and the dynamic stiffness method, and for multiple-connected regions using the dynamic stiffness method only. Results are presented for many cases and all are in excellent agreement with the published results in the literature.

TABLE 10

Values of Ω for three equal mass membranes

	$f_1(\xi) = 1 + \xi^2$	$f_2(\xi) = 1{\cdot}5$	$f_3(\xi) = 2 - \xi^2$
1	2·17358	1·96353	1·79950
2	4·84162	4·50712	4·25396
3	7·55950	7·06574	6·70441
4	10·28783	9·62775	9·15225
5	13·01987	12·19104	11·59875
6	15·75367	14·75496	14·04450
7	18·48844	17·31923	16·48980
8	21·22380	19·88371	18·93478
9	23·95956	22·44834	21·37936
10	26·69558	25·01294	23·80933

TABLE 11

Values of Ω for three section annular membranes

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
Ω_1	2·3820	2·8290	3·6944	4·6034
Ω_2	5·0887	5·4246	6·2201	7·1584
Ω_3	7·8720	8·1403	8·8303	9·7092

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